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A Comitant Curve of the Plane Quartic.[*]

BY TERESA COHEN.

I. THE CONIC ON A LINE AND THE FOUR TANGENTS AT ITS INTERSECTIONS WITH A PLANE QUARTIC.

If a quartic curve be cut by a line ξ and the tangents to the quartic at the intersections be drawn, the four tangents together with ξ itself determine a conic. This conic is to be expressed in terms of known comitants of the quartic.

For this it is necessary to know the degree of the conic both in ξ and in the coefficients of the quartic. While not known directly, it may be obtained indirectly by finding the contact of ξ with the conic. A line conic may be considered as the locus of lines cutting four fixed lines in a given double ratio. Now suppose two lines ξ and ξ' to cut the quartic in A, B, C, D and A', B', C', D', respectively, so that

$$(AB/CD)=(A'B'/C'D'). \tag{1}$$

Then the lines $AA', BB', CC', DD', \xi, \xi'$ touch a conic. Now allow ξ' to approach ξ, $(A'B'/C'D')$ being kept constantly equal to (AB/CD). AA', BB', CC', DD' approach the tangents to the quartic at A, B, C, D, respectively, the conic approaches the required conic, and the intersection of ξ and ξ' approaches the contact of ξ with the latter. But ξ and ξ', cutting the quartic in a constant double ratio, are lines of a known locus. Therefore, the contact of ξ with the required conic is the same as its contact with a known curve.

A quartic $(\alpha x)^4$ has two contravariants, $(s\xi)^4$ and $(t\xi)^6$, the loci of lines cut by the quartic in a self-apolar set and in harmonic pairs, respectively.[†] The locus of lines cutting the quartic in a given double ratio is

$$(s\xi)^4(s'\xi)^4(s''\xi)^4-\lambda(t\xi)^6(t'\xi)^6=0,$$

or, more briefly,

$$s^3-\lambda t^2=0,$$

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[*] Presented to the American Mathematical Society, December 28, 1916.

[†] These are understood to be taken as given in Salmon, "Higher Plane Curves," so that in symbolic notation $(s\xi)^4=\frac{1}{2}|\alpha\beta\xi|^2$ and $(t\xi)^6=\frac{1}{6}|\beta\gamma\xi|^2|\gamma\alpha\xi|^2|\alpha\beta\xi|^2$.

where λ is a constant depending on the given double ratio. The polar point of ξ with respect to this curve is

$$(s\xi)^3(s\eta)(s'\xi)^4(s''\xi)^4-\lambda(t\xi)^5(t\eta)(t'\xi)^6=0,$$

or

$$s^2s_1-\lambda tt_1=0.$$

If ξ is on the curve, we may eliminate λ between these two equations. Throwing out the factor s^2t, since ξ is in general on neither s nor t, we obtain for the polar point, or point of contact, of ξ

$$(s\xi)^3(s\eta)(t\xi)^6-(s\xi)^4(t\xi)^5(t\eta)=0, \text{ or } s_1t-st_1=0.$$

This is of degree 5 in the coefficients of the quartic and 9 in ξ; therefore, the conic itself should be of degree 5 in the coefficients of the quartic and 8 in ξ, which may be stated symbolically by saying that the conic is an $A^5\xi^8\eta^2$.

Under certain circumstances the degree in ξ will be lowered. The fact that the polar of ξ is s_1t-st_1 indicates that if $(s\xi)^4$ or $(t\xi)^6$ has a repeated factor, this factor appears in the $A^5\xi^8\eta^2$. Also suppose the curve to have a double point and ξ to be a line through it. Then ξ intersects the quartic in points whose double ratio is 0, 1, ∞ according to the order in which they are taken. But since any line through the double point can be regarded as an improper sort of tangent, the question arises, What lines should be taken as the two tangents here? Since with this choice of ξ in equation (1) $A=B$, then $A'=B'$. Then AA' is the same line as BB' and since the tangents at A and B are the limiting positions of these as ξ' approaches ξ, then the two tangents at the double point are to be taken as coincident. But then any line η cuts the four tangents where ξ meets the quartic in the same double ratio as ξ does and the conic becomes arbitrary. Therefore, the double points factor out of the $A^5\xi^8\eta^2$ at least once. The degree in ξ is not high enough for them to occur more than once.

There are only five independent forms of proper order to make up the $A^5\xi^8\eta^2$. Therefore,

$$\begin{aligned}A^5\xi^8\eta^2=\lambda(s\xi)^2(s\eta)^2(t\xi)^6+\mu(s\xi)^3(s\eta)(t\xi)^5(t\eta)+\nu(s\xi)^4(t\xi)^4(t\eta)^2\\+\rho(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)|\alpha\xi\eta|^2+\sigma(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\xi\eta|^2,\end{aligned}\tag{2}$$

where λ, μ, ν, ρ, σ are constant coefficients, the value of which is to be determined. Since the conic is on ξ,

$$\lambda(s\xi)^4(t\xi)^6+\mu(s\xi)^4(t\xi)^6+\nu(s\xi)^4(t\xi)^6=0,$$

$$\lambda+\mu+\nu=0.\tag{I}$$

This is one relation on the coefficients. For further relations special quartics for which the conic is known must be used.

Special Cases.

A. The simplest quartic for which the conic is known is $4x_0(x_1^3+x_2^3)=0$, representing four lines, three of which pass through the point $(1, 0, 0)$. The conic for any line ξ is made up of this point and the point where ξ cuts the fourth line, $x_0=0$. Its equation is

$$\eta_0(\eta_1\xi_2-\eta_2\xi_1)=0.$$

The double points of the quartic and the repeated factors of $(s\xi)^4$ and $(t\xi)^6$ account for an outstanding factor of degree 7 in ξ. Therefore,

$$A^5\xi^8\eta^2=\xi_0^4(\xi_1^3-\xi_2^3)\,(\eta_0\eta_1\xi_2-\eta_0\eta_2\xi_1).$$

By substitution in equation (2),

$$\begin{aligned}A^5\xi^8\eta^2=\eta_0^2(\xi_0^3\xi_1^4\xi_2+\xi_0^3\xi_1\xi_2^4)\,(8\lambda+12\mu+\tfrac{48}{5}\nu+36\rho+48\sigma)\\+\overset{2}{\Sigma}\eta_0\eta_1[\xi_0^4\xi_1^3\xi_2(16\lambda+18\mu+\tfrac{144}{5}\nu-54\rho-96\sigma)+\xi_0^4\xi_2^4(16\lambda+6\mu-18\rho)]\\+\overset{2}{\Sigma}\eta_1^2\cdot\xi_0^5\xi_1^2\xi_2(6\mu+\tfrac{48}{5}\nu+18\rho+48\sigma)+\eta_1\eta_2(\xi_0^5\xi_1^3+\xi_0^5\xi_2^3)\,(8\lambda+6\mu+18\rho).\end{aligned}$$

From this come the relations

$$4\lambda+3\mu+9\rho=0, \tag{II}$$

and

$$\mu+\frac{8}{5}\nu+3\rho+8\sigma=0. \tag{III}$$

B. The same relations would have been obtained by use of the four general lines

$$3\,[x_0^4+x_1^4+x_2^4-2x_1^2x_2^2-2x_2^2x_0^2-2x_0^2x_1^2]=0,$$

for which the conic is evidently

$$\begin{vmatrix}\eta_0^2 & \eta_1^2 & \eta_2^2\\ \xi_0^2 & \xi_1^2 & \xi_2^2\\ 1 & 1 & 1\end{vmatrix}=0.$$

By including the six double points

$$\begin{aligned}A^5\xi^8\eta^2=(\xi_1^2-\xi_2^2)\,(\xi_2^2-\xi_0^2)\,(\xi_0^2-\xi_1^2)\overset{3}{\Sigma}\eta_0^2(\xi_1^2-\xi_2^2)\\=\overset{3}{\Sigma}\eta_0^2(-\xi_0^4\xi_1^4+2\xi_0^4\xi_1^2\xi_2^2-\xi_0^4\xi_2^4+\xi_0^2\xi_1^6-\xi_0^2\xi_1^4\xi_2^2-\xi_0^2\xi_1^2\xi_2^4+\xi_0^2\xi_2^6-\xi_1^6\xi_2^2+2\xi_1^4\xi_2^4-\xi_1^2\xi_2^6).\end{aligned}$$

C. The lemniscate

$$6\,(x_1^2x_2^2+x_2^2x_0^2+x_0^2x_1^2)=0$$

is another case for which the conic can be found. Since it has three double points, $\xi_0\xi_1\xi_2$, and $(s\xi)^4=3(\xi^2)^2$, where $(\xi^2)=\xi_0^2+\xi_1^2+\xi_2^2$, the $A^5\xi^8\eta^2$ has the factor $\xi_0\xi_1\xi_2(\xi^2)$ and the conic reduces to the third degree in ξ.

That the conic can be found in this case is due to the property of the lemniscate that a line satisfying the equation $(\xi^2)=0$ cuts the lemniscate so that the four tangents at the intersections meet in the point $(\eta/\xi)=0$, or $(\eta_0\xi_1\xi_2)=0$. The conic, then, breaks up into two points, of which this is one; the other, found from s_1t-st_1, is $(\xi\eta)=0$. Therefore, for such lines as obey the relation $(\xi^2)=0$ the conic is

$$(\xi\eta)(\eta_0\xi_1\xi_2)=0.$$

But the conic in general can differ from this only by terms containing (ξ^2), and must be of the form

$$(\xi\eta)(\eta_0\xi_1\xi_2)+\varkappa(\xi^2)(\eta_1\eta_2\xi_0)=0.$$

Since it must be on ξ, $\varkappa=-1$. Therefore, the conic is

$$(\xi\eta)(\eta_0\xi_1\xi_2)-(\xi^2)(\eta_1\eta_2\xi_0)=0, \text{ or } \xi_0\xi_1\xi_2(\eta^2)-(\eta_1\eta_2\xi_0^3)=0.$$

It can furthermore be shown by direct proof that this conic touches the four tangents at the intersections of ξ with the quartic. Let y be one of these points. Then

$$(\xi y)=0, \quad \left(\frac{1}{y^2}\right)=0.$$

The tangent at y is $(x/y^3)=0$; that is, its coordinates are $1/y_0^3$, $1/y_1^3$, $1/y_2^3$. Substituting these values in the equation of the conic and remembering that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$, we find

$$\xi_0\xi_1\xi_2\left(\frac{1}{y^6}\right)-\left(\frac{\xi_0^3}{y_1^3y_2^3}\right)=\frac{3\,\xi_0\xi_1\xi_2}{y_0^2y_1^2y_2^2}-\frac{(\xi^3y^3)}{y_0^3y_1^3y_2^3}=\frac{3\,\xi_0\xi_1\xi_2\,y_0y_1y_2-3\,\xi_0\xi_1\xi_2\,y_0y_1y_2}{y_0^8y_1^3y_2^3}=0.$$

Therefore, the tangent to the quartic at y is a line of the conic.

To obtain the $A^5\xi^8\eta^2$ the conic must be multiplied by the outstanding factor $(\xi^2)\xi_0\xi_1\xi_2$.

$$A^5\xi^8\eta^2=\overset{3}{\Sigma}\eta_0(\xi_0^4\xi_1^2\xi_2^2+\xi_0^2\xi_1^4\xi_2^2+\xi_0^2\xi_1^2\xi_2^4)+\overset{3}{\Sigma}\eta_1\eta_2\xi_1\xi_2(-\xi_0^6-\xi_0^4\xi_1^2-\xi_0^4\xi_2^2).$$

The relations on λ, μ, ν, ρ, σ, given by this special qnartic, however, are all more simply found from the following case.

D. The type of the conic for the quartic $x_0^4+x_1^4+x_2^4=0$ can be obtained from consideration of

$$s_1t-st_1=\xi_0\xi_1\xi_2[(\xi^3\eta)\xi_0\xi_1\xi_2-\tfrac{1}{3}(\xi^4)(\eta_0\xi_1\xi_2)].$$

To give such a polar as to ξ (the repeated factor $\xi_0\xi_1\xi_2$ of $(t\xi)^6$ being dropped) the conic must be of the form

$$l(\xi^2\eta^2)\xi_0\xi_1\xi_2+m(\xi^3\eta)(\eta_0\xi_1\xi_2)+n(\xi^4)(\eta_1\eta_2\xi_0),$$

where l, m, n are to be determined. Since the conic is on ξ,

$$l+3m+3n=0.$$

For further relations on l, m, n take a special line, say through the reference point $\xi_0=0$. The line is then of the form $\varkappa x_1+x_2=0$, and it meets the curve in four points given by

$$x_0^4+(1+\varkappa^4)x_1^4=0.$$

Let α be any one of the fourth roots of $-(1+\varkappa)^4$. Then the four points are

$$(\alpha, 1, -\varkappa),\ (-\alpha, 1, -\varkappa),\ (i\alpha, 1, -\varkappa),\ (-i\alpha, 1, -\varkappa);$$

the four tangents to the curve at these points are

$$\alpha^3x_0+x_1-\varkappa^3x_2=0,\ -\alpha^3x_0+x_1-\varkappa^3x_2=0,\ -i\alpha^3x_0+x_1-\varkappa^3x^4=0,\ i\alpha^3x_0+x_1-\varkappa^3x_2=0.$$

These are all satisfied by the point $(0, \varkappa^3, 1)$; for the line $\varkappa x_1+x_2=0$, $s_1t-st_1=\eta_0$. Then for the line $(0, \varkappa, 1)$

$$m(\varkappa^3\eta_1+\eta_2)\cdot\varkappa\eta_0+n(\varkappa^4+1)(\eta_0\eta_1+\varkappa\eta_0\eta_2)\equiv\eta_0(\varkappa^3\eta_1+\eta_2).$$
$$\therefore\ n=0,\ m=-1,\ l=3.$$

The conic is

$$3(\xi^2\eta^2)\xi_0\xi_1\xi_2-(\xi^3\eta)(\eta_0\xi_1\xi_2)=0.$$

By a method similar to that for the lemniscate it can be shown satisfied by $(xy^3)=0$, the tangent at y, when $(y^4)=0$, $(\xi y)=0$. To obtain the $A^5\xi^8\eta^2$ multiply by $\xi_0\xi_1\xi_2$, the repeated factor of $(t\xi)^6$.

$$A^5\xi^8\eta^2=\overset{3}{\Sigma}\eta_0^2\cdot 2\xi_0^4\xi_1^2\xi_2^2+\overset{3}{\Sigma}\eta_1\eta_2\xi_1\xi_2(-\xi_0^2\xi_1^4-\xi_0^2\xi_2^4).$$

Substituting in (2),

$$A^5\xi^8\eta^2=\overset{3}{\Sigma}\eta_0^2[\xi_0^4\xi_1^2\xi_2^2(\lambda+\tfrac{1}{3}\mu+\tfrac{1}{15}\nu+2\sigma)+(\xi_1^6\xi_2^2+\xi_1^2\xi_2^6)(\tfrac{1}{15}\nu+\rho+2\sigma)]$$
$$+\overset{3}{\Sigma}\eta_1\eta_2\xi_1\xi_2[(\xi_0^2\xi_1^4+\xi_0^2\xi_2^4)(\tfrac{1}{3}\mu+\tfrac{4}{15}\nu-2\sigma)+\xi_0^6(\tfrac{4}{15}\nu-2\rho-2\sigma)].$$

Then $$\tfrac{1}{15}\nu+\rho+2\sigma=0. \qquad \text{(IV)}$$

This equation together with (I), (II) and (III) is sufficient to determine $\lambda, \mu, \nu, \rho, \sigma$ (or rather their ratios).

$$\lambda=0,\ \mu=15,\ \nu=-15,\ \rho=-5,\ \sigma=3.$$

E. The formula

$$A^5\xi^8\eta^2=15(s\xi)^3(s\eta)(t\xi)^5(t\eta)-15(s\xi)^4(t\xi)^4(t\eta)^2-5(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)|\alpha\xi\eta|^2$$
$$+3(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\xi\eta|^2$$

may be tested on the general quartic

$$ax_0^4+4a_1x_0^3x_1+4a_2x_0^3x_2+6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3+12mx_0x_1^2x_2$$
$$+12nx_0x_1x_2^2+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4=0 \qquad (3)$$

for a special reference triangle and line ξ. Let the line ξ be $x_0=0$ and let two of the tangents at its intersections be $x_1=0$ and $x_2=0$. This requires that

$$b=c=b_0=c_0=0.$$

Then beside the two reference points $(0,1,0)$ and $(0,0,1)$ the line $x_0=0$ cuts the quartic in two points given by

$$4b_2x_1^2+6fx_1x_2+4c_1x_2^2=0. \tag{4}$$

Let these two points be $(0,A,1)$ and $(0,B,1)$. Then the tangent at $(0,A,1)$ is

$$x_0(3mA^2+3nA)+x_1(3b_2A^2+3fA+c_1)+x_2(b_2A^3+3fA^2+3c_1A)=0,$$

or, making use of the fact that $(0,A,1)$ must satisfy (4),

$$x_0(3mA+3n)+x_1(2b_2A+\tfrac{3}{2}f)+x_2(\tfrac{3}{2}fA+2c_1)=0. \tag{5}$$

Similarly, the tangent at $(0,B,1)$ is

$$x_0(3mB+3n)+x_1(2b_2B+\tfrac{3}{2}f)+x_2(\tfrac{3}{2}fB+2c_1)=0. \tag{6}$$

The conic, being on the reference lines, must be of the form

$$\alpha_0\eta_1\eta_2+\alpha_1\eta_2\eta_0+\alpha_2\eta_0\eta_1=0.$$

Requiring it to be satisfied by the lines (5) and (6) gives two homogeneous equations in α_0, α_1, α_2, which can be solved for them in terms of A and B. But A and B enter symmetrically, so that they can be gotten rid of by use of symmetric functions of the roots of (4). The conic is obtained in the form

$$\eta_0\eta_1(-4b_2c_1n+3c_1fm)+\eta_0\eta_2(-4b_2c_1m+3b_2fn)+\eta_1\eta_2(6b_2n^2+6c_1m^2-9fmn)=0,$$

when the factor $-16b_2c_1+9f^2$ has been dropped.

By use of equation (2)*

$$\begin{aligned} A^5\xi^8\eta^2 = {} & \eta_0^2[b_2^2c_1^2f(-8\mu-8\nu\equiv0) \\ & \quad +b_2c_1f^3(10\mu+10\nu\equiv0) \\ & \quad +f^5(-3\mu-3\nu\equiv0)] \\ & +\sum^{2}2\eta_0\eta_1[b_2^2c_1^2n(\tfrac{4}{3}\mu+\tfrac{8}{3}\nu\equiv-20) \\ & \qquad +b_2c_1^2fm(5\mu+4\nu\equiv15) \\ & \qquad +b_2c_1f^2n(-6\mu-6\nu\equiv0) \\ & \qquad +c_1f^3m(-3\mu-3\nu\equiv0) \\ & \qquad +f^4n(3\mu+3\nu\equiv0)] \end{aligned}$$

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* For the coefficients of s and t see the note at the end of the article.

$$
\begin{aligned}
&+\sum^{2}\eta_1^2[b_2^2c_1^2g(-\tfrac{8}{15}\nu+16\rho+24\sigma\equiv 0)\\
&\qquad+b_2c_1^3h(\tfrac{24}{15}\nu+8\sigma\equiv 0)\\
&\qquad+b_2c_1^2fl(-\tfrac{24}{15}\iota-8\sigma\equiv 0)\\
&\qquad+b_2c_1^2mn(-2\mu-\tfrac{48}{15}\nu-6\rho-16\sigma\equiv 0)\\
&\qquad+b_2c_1f^2g(\tfrac{18}{15}\nu-24\rho-34\sigma\equiv 0)\\
&\qquad+c_1^2f^2h(-\tfrac{18}{15}\nu-6\sigma\equiv 0)\\
&\qquad+b_2c_1fn^2(2\mu+\tfrac{48}{15}\nu+6\rho+16\sigma\equiv 0)\\
&\qquad+c_1^2fm^2(-3\mu-9\rho\equiv 0)\\
&\qquad+c_1f^3l(\tfrac{18}{15}\nu+6\sigma\equiv 0)\\
&\qquad+c_1f^2mn(6\mu+\tfrac{36}{15}\nu+18\rho+12\sigma\equiv 0)\\
&\qquad+f^4g(-\tfrac{9}{15}\nu+9\rho+12\sigma\equiv 0)\\
&\qquad+f^3n^2(-3\mu-\tfrac{36}{15}\nu-9\rho-12\sigma\equiv 0)]\\
&+2\eta_1\eta_2[b_2^2c_1^2l(-\tfrac{16}{15}\nu-16\rho-32\sigma\equiv 0)\\
&\qquad\quad+(b_2^2c_1fg+b_2c_1^2fh)(-\tfrac{12}{15}\nu-4\sigma\equiv 0)\\
&\qquad\quad+(b_2^2c_1n^2+b_2c_1^2m^2)(-\mu-\tfrac{24}{15}\nu+15\rho+32\sigma\equiv 30)\\
&\qquad\quad+b_2c_1f^2l(\tfrac{36}{15}\nu+24\rho+52\sigma\equiv 0)\\
&\qquad\quad+b_2c_1fmn(-\mu+\tfrac{12}{15}\nu-30\rho-56\sigma=-45)\\
&\qquad\quad+(b_2f^3g+c_1f^3h)(\tfrac{9}{15}\nu+3\sigma\equiv 0)\\
&\qquad\quad+(b_2f^2n^2+c_1f^2m^2)(3\mu+\tfrac{18}{15}\nu-9\rho-24\sigma\equiv 0)\\
&\qquad\quad+f^4l(-\tfrac{18}{15}\nu-9\rho-21\sigma\equiv 0)\\
&\qquad\quad+f^3mn(-3\mu-\tfrac{9}{15}\nu+18\rho+42\sigma\equiv 0)]\\
&=5b_2c_1[\sum^{2}2\eta_0\eta_1(-4b_2c_1n+3c_1fm)+2\eta_1\eta_2(6b_2n^2+6c_1m^2-9fmn)].
\end{aligned}
$$

Therefore, *the conic on a line ξ and the tangents at its four intersections with a quartic is given by*

$$
\begin{aligned}
15(s\xi)^3(s\eta)(t\xi)^5(t\eta)-15(s\xi)^4(t\xi)^4(t\eta)^2-5(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)\,|\alpha\xi\eta|^2\\
+3(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\xi\eta|^2.
\end{aligned}
$$

II. THE LOCUS OF LINES MEETING THE QUARTIC SO THAT THREE OF THE TANGENTS AT THE INTERSECTIONS ARE ON A POINT.

For certain lines ξ the conic breaks up. These are the lines such that three of the tangents at the intersections with the quartic are on a point. The discriminant of the $A^5\xi^8\eta^2$, an $A^{15}\xi^{24}$, is therefore the locus of such lines. The line equation of the quartic does not enter into this $A^{15}\xi^{24}$, as may be seen from case E above, where $16\,b_2c_1-9\,f^2$, the condition that $x_0=0$ be tangent to the quartic, is not a factor in the discriminant of the conic.

The expression obtained for the $A^5\xi^8\eta^2$ is of the form

$$
\begin{aligned}
&(a_1x_0^2+b_1x_1^2+c_1x_2^2+2f_1x_1x_2+2g_1x_2x_0+2h_1x_0x_1)\\
+\,&(a_2x_0^2+b_2x_1^2+c_2x_2^2+2f_2x_1x_2+2g_2x_2x_0+2h_2x_0x_1)\\
+\,&(a_3x_0^2+b_3x_1^2+c_3x_2^2+2f_3x_1x_2+2g_3x_2x_0+2h_3x_0x_1)\\
+\,&(a_4x_0^2+b_4x_1^2+c_4x_2^2+2f_4x_1x_2+2g_4x_2x_0+2h_4x_0x_1).
\end{aligned}
$$

Therefore, if

$$
\Delta_{iii}=\begin{vmatrix} a_i & h_i & g_i\\ h_i & b_i & f_i\\ g_i & f_i & c_i\end{vmatrix},\quad
\Delta_{iij}=\sum^{3}\begin{vmatrix} a_i & h_i & g_j\\ h_i & b_i & f_j\\ g_i & f_i & c_j\end{vmatrix},\quad
\Delta_{ijk}=\sum^{6}\begin{vmatrix} a_i & h_j & g_k\\ h_i & b_j & f_k\\ g_i & f_j & c_k\end{vmatrix},
$$

then

$$
\begin{aligned}
A^{15}\xi^{24}&=\begin{vmatrix} a_1+a_2+a_3+a_4 & h_1+h_2+h_3+h_4 & g_1+g_2+g_3+g_4\\ h_1+h_2+h_3+h_4 & b_1+b_2+b_3+b_4 & f_1+f_2+f_3+f_4\\ g_1+g_2+g_3+g_4 & f_1+f_2+f_3+f_4 & c_1+c_2+c_3+c_4\end{vmatrix}\\
&=\sum^{4}\Delta_{111}+\sum^{12}\Delta_{112}+\sum^{4}\Delta_{123},
\end{aligned}
$$

where

$$
\begin{aligned}
&\Delta_{111}=0,\quad \Delta_{333}=0,\quad \Delta_{444}=0,\\
&\Delta_{222}=-\tfrac{15^3}{6}(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2,\\
&\Delta_{112}=-\tfrac{15^3}{24}\{(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2\\
&\qquad\qquad -3(s\xi)^4(s'\xi)^3(s''\xi)^3(t\xi)^6(t'\xi)^4(t''\xi)^4|s't't''|\,|s''t't''|\},\\
&\Delta_{122}=\tfrac{15^3}{6}(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2,\\
&\Delta_{113}=\tfrac{15^2.5}{4}(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)\{(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^6\\
&\qquad\qquad -2(s''\xi)^4(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^5(t'\alpha)\\
&\qquad\qquad +(s''\xi)^4(s'''\xi)^4(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\\
&\Delta_{133}=\tfrac{15\cdot5^2}{2}(s\xi)^4(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha')(s^{\text{iv}}\xi)^3(s^{\text{iv}}\alpha')|\alpha\alpha'\xi|^2(t\xi)^6,\\
&\Delta_{114}=-\tfrac{15^2.3}{4}(s\xi)^4(s'\xi)^2(s'\alpha)^2\{(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^6\\
&\qquad\qquad -2(s''\xi)^4(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^5(t'\alpha)\\
&\qquad\qquad +(s''\xi)^4(s'''\xi)^4(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\\
&\Delta_{144}=\tfrac{15\cdot3^2}{2}(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2(s^{\text{iv}}\xi)^2(s^{\text{iv}}\alpha')^2|\alpha\alpha'\xi|^2(t\xi)^6,\\
&\Delta_{223}=-15^2.5(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)\\
&\qquad\qquad \{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\\
&\Delta_{233}=-\tfrac{15\cdot5^2}{2}(s\xi)^4(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha')(s^{\text{iv}}\xi)^3(s^{\text{iv}}\alpha')|\alpha\alpha'\xi|^2(t\xi)^6,\\
&\Delta_{224}=15^2.3(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2\\
&\qquad\qquad \{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\\
&\Delta_{244}=-\tfrac{15\cdot3^2}{2}(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2(s^{\text{iv}}\xi)^2(s^{\text{iv}}\alpha')^2|\alpha\alpha'\xi|^2(t\xi)^6,\\
&\Delta_{334}=0,\quad \Delta_{344}=0,\\
&\Delta_{123}=15^2.5(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)\\
&\qquad\qquad \{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},\\
&\Delta_{124}=-15^2.3(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2\\
&\qquad\qquad \{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},
\end{aligned}
$$

$$\Delta_{134} = -15^2(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)\,(s'''\xi)^3(s'''\alpha)\,(s^{\text{iv}}\xi)^2(s^{\text{iv}}\alpha)^2|\alpha\alpha'\xi|^2(t\xi)^6,$$
$$\Delta_{234} = 15^2(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)\,(s'''\xi)^3(s'''\alpha)\,(s^{\text{iv}}\xi)^2(s^{\text{iv}}\alpha)^2|\alpha\alpha'\xi|^2(t\xi)^6.$$

Then for convenience dropping the factor $-\frac{15^2}{8}$,

$$\begin{aligned} A^{15}\xi^{24} = {} & 5(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2 \\ & -15(s\xi)^4(s'\xi)^3(s''\xi)^3(t\xi)^6(t'\xi)^4(t''\xi)^4|s't't''|\,|s''t't''| \\ & -2\{5(s\xi)^3(s\alpha)\,(s'\xi)^3(s'\alpha) - 3(s\xi)^4(s'\xi)^2(s'\alpha)^2\} \\ & \quad \{(s''\xi)^3(s''\alpha)\,(s'''\xi)^3(s'''\alpha)\,(t\xi)^6(t'\xi)^6 \\ & \quad -2(s''\xi)^4(s'''\xi)^3(s'''\alpha)\,(t\xi)^6(t'\xi)^5(t'\alpha) \\ & \quad +(s''\xi)^4(s'''\xi)^4(t\xi)^5(t\alpha)\,(t'\xi)^5(t'\alpha)\}. \end{aligned}$$

Special Cases.

The correctness of this formula may be tested on the special cases given before for which the conic is known.

A. For the quartic $4x_0(x_1^3+x_2^3)=0$

$$A^{15}\xi^{24} \equiv 0.$$

B. For the four lines $3[x_0^4+x_1^4+x_2^4-2x_1^2x_2^2-2x_2^2x_0^2-2x_0^2x_1^2]=0$

$$A^{15}\xi^{24} = (\xi_1^2-\xi_2^2)^4(\xi_2^2-\xi_0^2)^4(\xi_0^2-\xi_1^2)^4.$$

C. For the lemniscate $6[x_1^2x_2^2+x_2^2x_0^2+x_0^2x_1^2]=0$

$$A^{15}\xi^{24} = 180\cdot 162^2(\xi^2)^4\xi_0^4\xi_1^4\xi_2^4[(\xi^2)^2-3(\xi_1^2\xi_2^2)].$$

D. For the quartic $x_0^4+x_1^4+x_2^4=0$

$$A^{15}\xi^{24} = \tfrac{10}{3}\,\xi_0^4\xi_1^4\xi_2^4[(\xi^4)\,(\xi_1^4\xi_2^4)-9\xi_0^4\xi_1^4\xi_2^4].$$

In each case the result obtained by the formula agrees with that obtained directly from the $A^5\xi^8\eta^2$.

It should be noted that the double points and repeated factors of s and t, which occur *once* in the $A^5\xi^8\eta^2$ and therefore would be expected to enter the $A^{15}\xi^{24}$ *three* times, actually occur *four* times.

Singular Lines of the Locus.

Of special interest in connection with the $A^{15}\xi^{24}$ are the lines ξ such that

$$(s\xi)^3(s\eta)\,(t\xi)^6-(s\xi)^4(t\xi)^5(t\eta)\equiv 0;$$

that is, the contact of ξ with the $A^5\xi^8\eta^2$ is indeterminate. This occurs 1) if s or t has a repeated factor; 2) for the common lines of s and t, the stationary lines of the quartic; 3) for the lines having the same polar point as to s and t.

1) For the general quartic neither s nor t has repeated factors. Therefore, this case need not be discussed here.

2) It is evident that the common lines of s and t are double lines of the

$A^{15}\xi^{24}$, for every term of the latter contains either s^2, st, or t^2. Furthermore, the polar conic when $s=t=0$ becomes

$$\begin{aligned}&(s\xi)^3(s\eta)(s'\xi)^3(s'\eta)\cdot -\tfrac{20}{3}(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\\ &+(s\xi)^3(s\eta)(t\xi)^5(t\eta)[20(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t'\xi)^5(t'\alpha)\\ &\qquad\qquad -15(s'\xi)^3(s''\xi)^3(t'\xi)^4(t''\xi)^4|s't't''|\,|s''t't''|\,]\\ &+(t\xi)^5(t\eta)(t'\xi)^5(t'\eta)\cdot -15(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha).\end{aligned}$$

But $(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)=0$ is the condition that the point $(s\xi)^3(s\eta)$ lie on the polar conic as to the quartic of $(t\xi)^5(t\eta)$, a condition known to be fulfilled when ξ is a stationary line of the quartic, since $(t\xi)^5(t\eta)$ is the flex itself. Therefore, the coefficient of $(s\xi)^3(s\eta)(s'\xi)^3(s'\eta)$ vanishes, showing that $(t\xi)^5(t\eta)$ is one of the points of contact of ξ with the $A^{15}\xi^{24}$. Therefore ξ is not only a double line of both the $A^{15}\xi^{24}$ and the quartic $(ax)^4$, but also has one point of contact the same in each case, so that the stationary lines account for $24(2\cdot 2+1)=120$ of the common lines of the two curves.

3) For a line ξ to have the same polar point as to s and t requires that

$$\frac{(s\xi)^3 s_0}{(t\xi)^5 t_0}=\frac{(s\xi)^3 s_1}{(t\xi)^5 t_1}=\frac{(s\xi)^3 s_2}{(t\xi)^5 t_2}.$$

These equations have $25+9+15=49$ solutions. Among these must be included the twenty-eight double lines of the quartic, for both $s_1t-st_1\equiv 0$ and $s^2s_1-27tt_1\equiv 0$ hold if $s^3-27t^2=0$, which is true for the double lines. The remaining twenty-one lines are those which meet the quartic so that all four tangents at the intersections are on a point; they correspond to the twenty-one double points of the Steinerian, which are points whose polar cubic is a conic and a line. For all of these lines the $A^5\xi^8\eta^2$ is degenerate, so that they are at least lines of the $A^{15}\xi^{24}$. But presumably they are more than this, and indeed the twenty-one lines are at least triple lines, for they cause the $A^5\xi^8\eta^2$ to vanish identically. If these lines are double lines, their polar points as to the $A^{15}\xi^{24}$ should vanish. But the polar point of ξ when $s_1t-st_1\equiv 0$ becomes

$$\begin{aligned}-\tfrac{5}{12}(s'\xi)^4(s''\xi)^4[\,&(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2(s\xi)^3(s\eta)\\ &-(s\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2(t\xi)^3(t\eta)\,],\end{aligned}$$

and it is not evident that this vanishes. Furthermore, though it can be shown that $(t'\xi)^4(t''\xi)^4|t't''x|^2$ is a numerical multiple of

$$\begin{aligned}&9(s\xi)^2(s\alpha)^2(\alpha x)^2(t\xi)^6-\tfrac{15}{4}(s\xi)^4(t\xi)^4(t\alpha)^2(\alpha x)^2\\ &\quad+10(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)|\alpha\alpha'\xi|^2(\alpha'x)^2-\tfrac{39}{8}(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\alpha'\xi|^2(\alpha x)^2\\ &+(x\xi)[-18(s\xi)^2(s\alpha)^2(t\xi)^5(t\alpha)(\alpha x)+\tfrac{75}{7}(s\xi)^4(t\xi)^3(t\alpha)^3(\alpha x)]\\ &+(x\xi)^2[\tfrac{17}{6}(s\alpha)^4(t\xi)^6-\tfrac{21}{2}(s\xi)^2(s\alpha)^2(t\xi)^4(t\alpha)^2+\tfrac{50}{7}(s\xi)^3(s\alpha)(t\xi)^3(t\alpha)^3\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad -\tfrac{145}{56}(s\xi)^4(t\xi)^2(t\alpha)^4],\end{aligned}$$

the substitution of this value does not give an expression that vanishes on its face.

Let us first consider the double lines for the general quartic (3) with a convenient reference triangle. Let $x_0=0$ be a double line with contacts $(0,1,0)$ and $(0,0,1)$, and let $(1,0,0)$ be the polar point of x_0 as to both s and t.* This reference scheme requires that in the equation of the quartic

$$b=c=b_2=c_1=m=n=0.$$

Then (giving only the highest powers of ξ_0)

$$\begin{aligned}A^5\xi^8\eta^2 = {} & \eta_0^2[270b_0c_0f^3\xi_0^6\xi_1\xi_2+\ldots.]+\eta_1^2[-90b_0c_0^2f^2\xi_0^7\xi_1+\ldots.]\\ & +\eta_2^2[-90b_0^2c_0f^2\xi_0^7\xi_2+\ldots.]\\ & +2\eta_1\eta_2[-45b_0c_0f^3\xi_0^8+135c_0f^3h\xi_0^7\xi_1+135b_0f^3g\xi_0^7\xi_2+\ldots.]\\ & +2\eta_0\eta_2[-45b_0c_0f^3\xi_0^7\xi_1+\ldots.]+2\eta_0\eta_1[-45b_0c_0f^3\xi_0^7\xi_2+\ldots.],\end{aligned}$$

so that

$$A^{15}\xi^{24} = -90b_0^3c_0^3f^9\xi_0^{22}\xi_1\xi_2 + \text{lower powers in } \xi_0.$$

Therefore, the double lines of the quartic are also double lines of the $A^{15}\xi^{24}$ with the same contacts. They then account for $28(2\cdot2+2)=168$ common lines of the two curves, and since it has already been shown that the stationary lines account for 120, all of the $24\times12=288$ common lines of the $A^{15}\xi^{24}$ and the quartic (in lines an $A^6\xi^{12}$) are accounted for.

For the twenty-one lines giving four tangents meeting in a point (which point is the common polar point of the line as to s and t) choose a reference scheme where $x_0=0$ is one of these lines with two of its intersections as $(0,1,0)$ and $(0,0,1)$ and let the tangents at these two points be $x_2=0$ and $x_1=0$, respectively, so that $(1,0,0)$ is the meeting point of the four tangents. Then

$$b=c=b_0=c_0=m=n=0.$$

$$\begin{aligned}A^5\xi^8\eta^2 = {} & \eta_0^2[\text{terms beginning with } \xi_0^6\xi_1^2]\\ & +\eta_1^2[\xi_0^6\xi_1\xi_2(120b_2^2c_1^2g-360b_2c_1^2gh-180b_2c_1fgl+270c_1f^2gh)\\ & \qquad +\xi_0^6\xi_2^2(240b_2^2c_1gl-90b_2^2fg^2-90b_2c_1fgh)+\ldots.]\\ & +\eta_2^2[\xi_0^6\xi_1^2(240b_2c_1^2hl-90b_2c_1fgh-90c_1^2fh^2)\\ & \qquad +\xi_0^6\xi_1\xi_2(-360b_2^2c_1gh+120b_2c_1^2h^2-180b_2c_1fhl+270b_2f^2gh)+\ldots.]\\ & 2\eta_1\eta_2[\xi_0^6\xi_1^2(30b_2^2c_1g^2-120b_2c_1^2gh+90c_1^3h^2+120b_2c_1^2l^2-90b_2c_1fgl\\ & \qquad -180c_1^2fhl+135c_1f^2gh)\\ & \qquad +\xi_0^6\xi_1\xi_2(-120b_2^2c_1gl-120b_2c_1^2hl-90b_2^2fg^2+540b_2c_1fgh\\ & \qquad -90c_1^2fh^2-180b_2c_1fl^2+270b_2f^2gl+27c_1f^2hl\\ & \qquad -261f^3gh-168f^3l^2)\\ & \qquad +\xi_0^6\xi_2^2(90b_2^3g^2-120b_2^2c_1gh+30b_2c_1^2h^2+120b_2^2c_1l^2-180b_2^2fgl\\ & \qquad -90b_2c_1fhl+135b_2f^2gh)+\ldots.]\\ & +2\eta_0\eta_2[\xi_0^7\xi_1(40b_2^2c_1^2l-15b_2^2c_1fg-15b_2c_1^2fh)\\ & \qquad +\xi_0^7\xi_2(-60b_2^3c_1g+20b_2^2c_1^2h-30b_2^2c_1fl+45b_2^2f^2g)+\ldots.]\\ & +2\eta_0\eta_1[\xi_0^7\xi_1(20b_2^2c_1^2g-60b_2c_1^3h-30b_2c_1^2fl+45c_1^2f^2h)\\ & \qquad +\xi_0^7\xi_2(40b_2^2c_1^2l-15b_2^2c_1fg-15b_2c_1^2fh)+\ldots.].\end{aligned}$$

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* It is evident that this polar point can not lie on the double line itself, for then the latter would have to be a line of both s and t, all of whose common lines are accounted for by the stationary lines of the quartic.

From this it is evident that ξ_0^{20} is the highest power of ξ_0 that can enter into the $A^{15}\xi^{24}$. Therefore, the twenty-one lines are quadruple lines of the locus. There is no reason for supposing that the $A^{15}\xi^{24}$ has any more singular lines.

NOTE.

$$(s\xi)^4 = A\xi_0^4+B\xi_1^4+C\xi_2^4+4A_1\xi_0^3\xi_1+4A_2\xi_0^3\xi_2+4B_0\xi_1^3\xi_0+4B_2\xi_1^3\xi_2+4C_0\xi_2^3\xi_0+4C_1\xi_2^3\xi_1$$
$$+6F\xi_1^2\xi_2^2+6G\xi_2^2\xi_0^2+6H\xi_0^2\xi_1^2+12L\xi_0^2\xi_1\xi_2+12M\xi_0\xi_1^2\xi_2+12N\xi_0\xi_1\xi_2^2,$$

where

$$\begin{aligned}
A &= bc-4b_2c_1+3f^2,\\
A_1 &= -cb_0+b_2c_0+3c_1m-3fn, \quad A_2 = -bc_0+b_0c_1+3b_2n-3fm,\\
F &= af-2a_1n-2a_2m+gh+2l^2,\\
L &= b_0c_0-b_2g-c_1h+2fl-mn.
\end{aligned}$$

$$(t\xi)^6 = \overset{3}{\Sigma}A_{600}\xi_0^6+6\overset{6}{\Sigma}A_{510}\xi_0^5\xi_1+15\overset{6}{\Sigma}A_{420}\xi_0^4\xi_1^2+30\overset{3}{\Sigma}A_{411}\xi_0^4\xi_1\xi_2+20\overset{3}{\Sigma}A_{330}\xi_0^3\xi_1^3$$
$$+60\overset{6}{\Sigma}A_{321}\xi_0^3\xi_1^2\xi_2+90A_{222}\xi_0^2\xi_1^2\xi_2^2,$$

where

$$\begin{aligned}
A_{600} &= bcf-bc_1^2-cb_2^2+2b_2c_1f-f^3,\\
A_{510} &= \tfrac{1}{3}[-bcn+bc_0c_1-2cb_0f+3cb_2m+2b_0c_1^2-b_2c_0f-2b_2c_1n-3c_1fm+3f^2n],\\
A_{420} &= \tfrac{1}{15}[bcg-bc_0^2+8cb_0n-6cb_2l+6cfh-9cm^2-8b_0c_0c_1+2b_2c_1g-6c_1^2h\\
&\qquad +4b_2c_0n+6c_2fm+6c_1fl+12c_1mn-3f^2g-12fn^2],\\
A_{411} &= \tfrac{1}{15}[2bcl-3bc_1g-3cb_2h+bc_0n+cb_0m+9b_0c_0f-10b_0c_1n-10b_2c_0m\\
&\qquad +4b_2c_1l+3b_2fg+3c_1fh+6b_2n^2+6c_1m^2-6f^2l-3fmn],\\
A_{330} &= \tfrac{1}{10}[ca_2b_2-2ca_1f-2cb_0g-6chn+9clm+2a_1c_1^2+2b_0c_0^2-a_2c_1f-b_2c_0g\\
&\qquad +6c_0c_1h-3c_0fl-3c_1gm-6c_0mn-6c_1ln+6fgn+4n^3],\\
A_{321} &= \tfrac{1}{30}[-bca_2+3ca_1b_2+bc_0g-5cb_0l+3chm-2a_2b_2c_1-3a_1c_1f+11b_0c_1g\\
&\qquad +8b_2c_0l-6b_0c_0n+3a_2f^2-15c_0fh+12c_1hn-9b_2gn+12c_0m^2-18c_1lm\\
&\qquad -3fgm+15fln-6mn^2],\\
A_{222} &= \tfrac{1}{90}[abc+2ab_2c_1+2ba_2c_0+2ca_1b_0-3af^2-3bg^2-3ch^2-20a_1b_2c_0\\
&\qquad -20a_2b_0c_1+24a_1c_1m+24a_2b_2m+24b_0c_0l-6a_1fn-6a_2fm-6b_0gn\\
&\qquad -6b_2gl-6c_0hm-6c_1hl+48fgh-30fl^2-30gm^2-30hn^2+48lmn].
\end{aligned}$$

All the other coefficients can be found by symmetry from those given here.

JOHNS HOPKINS UNIVERSITY, *October*, 1916.